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Oscillatory and Nonoscillatory Behavior of Second-Order Neutral Delay Difference Equations II

R. P. AGARWAL
Department of Mathematics, National University of Singapore
Kent Ridge, Singapore 0511

M. M. S. MANUEL AND E. THANDAPANI
Department of Mathematics, Madras University, P. G. Centre
Salem 636011, Tamil Nadu, India

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Abstract—For the difference equation

$$\Delta(p_n\Delta(y_n + h_ny_{n-k})) + q_{n+1}f(y_{n+1-\ell}) = 0, \qquad n \in Z = \{0, 1, \ldots\},$$

the existence of solutions in the classes M^+, M^-, and WOS is established.

Keywords—Neutral delay difference equations, Oscillatory solutions, Weakly oscillatory solutions.

1. INTRODUCTION

This is in continuation with our study [1] of the neutral difference equation

$$\Delta(p_n\Delta(y_n + h_ny_{n-k})) + q_{n+1}f(y_{n+1-\ell}) = 0, \qquad n \in Z = \{0, 1, \ldots\}, \tag{1}$$

where $\Delta y_n = y_{n+1} - y_n$, k, ℓ are fixed nonnegative integers, $\{p_n\}$, $\{h_n\}$, $\{q_n\}$ are real sequences, and

(c$_1$) $p_n > 0$ for all $n \in Z$ and q_n is not identically zero for large n,

(c$_2$) $f : \Re \to \Re$ is continuous and $uf(u) > 0$ for $u \neq 0$, and

(c$_3$) there exists a nonnegative function g such that

$$f(u) - f(v) = g(u,v)(u - v), \qquad \text{for all } u \neq v.$$

Let $m = \max\{k, \ell\}$ and N_0 be a fixed nonnegative integer. By a solution of (1), we mean a real sequence $\{y_n\}$ which is defined for all $n \geq N_0 - m$ and satisfies (1) for $n \geq N_0$. A solution $\{y_n\}$ of (1) is said to be nonoscillatory if all the terms y_n are eventually of fixed sign. Otherwise, the solution $\{y_n\}$ is called oscillatory. A nonoscillatory solution $\{y_n\}$ of (1) is said to be weakly oscillatory if $\{\Delta y_n\}$ changes sign for arbitrarily large values of n.

In [1], the set S of all nontrivial solutions of (1) has been divided into the following four classes:

$M^+ = \{\{y_n\} \in S :$ there exists an integer $N \in Z$ such that $y_n\Delta y_n \geq 0$ for all $n \geq N\}$;

$M^- = \{\{y_n\} \in S : \{y_n\}$ is nonoscillatory and there exists an integer $N \in Z$ such that $y_n\Delta y_n \leq 0$ for all $n \geq N\}$;

$$\begin{aligned} \mathrm{OS} &= \{\{y_n\} \in S : \text{for every integer } N \in Z, \text{ there exists } n \geq N \text{ such that } y_n y_{n+1} \leq 0\}; \\ \mathrm{WOS} &= \{\{y_n\} \in S : \{y_n\} \text{ is nonoscillatory and for every } N \in Z, \text{ there exists } n \geq N \text{ such that} \\ &\quad \Delta y_n \Delta y_{n+1} \leq 0\}. \end{aligned}$$

In fact, to establish oscillatory behavior of solutions of (1), in [1] we have provided sufficient conditions which ensure that the classes M^+, M^-, and WOS are empty. The purpose of this paper is to prove the existence of solutions of (1) in these classes.

2. EXISTENCE OF SOLUTIONS IN M^+, M^-, AND WOS

THEOREM 1. *With respect to the difference equation (1), assume that the following hold:*

(i) k *is an even positive integer,*

(ii) $h_n \equiv h \geq 0$ *and* $\neq 1$ *for all* $n \in Z$,

(iii) $q_n > 0$ *for all large values of* n,

(iv) $\sum_{n=n_0}^{\infty}(1/p_n) = \infty$, *and*

(v) $\sum_{n=n_0}^{\infty}(1/p_n)\sum_{s=n}^{\infty} q_{s+1} < \infty$.

Then, $M^+ \neq \emptyset$.

PROOF. We shall prove the existence of positive solutions of (1) in the class M^+; the existence of negative solutions in M^+ is similar.

First, assume that $0 \leq h < 1$. Let $K = \max\{|f(u)| : 3(1-h) \leq u \leq 4\}$. Choose a positive integer $N \geq n_0$ sufficiently large, so that

$$K \sum_{n=N}^{\infty} \frac{1}{p_n} \sum_{s=n}^{\infty} q_{s+1} \leq \frac{1-h}{4}.$$

Consider the Banach space B_N of all real sequences $y = \{y_n\}$, $n \geq N - m$ with the sup norm $\|y\| = \sup_{n \geq N-m} |y_n|$. We define a subset $\mathcal{S}$ of B_N as

$$\mathcal{S} = \{y \in B_N : 3(1-h) \leq y_n \leq 4,\ n \geq N - m\}.$$

Clearly, $\mathcal{S}$ is a bounded, closed, and convex subset of B_N. Now, we define an operator $T : \mathcal{S} \to B_N$ as follows:

$$Ty_n = \begin{cases} 3 + h - hy_{n-k} + \displaystyle\sum_{s=N}^{n-1} \frac{1}{p_s} \sum_{\eta=s}^{\infty} q_{\eta+1} f(y_{\eta+1-\ell}), & n \geq N, \\ Ty_N = 3 + h - hy_{N-k}, & N - m \leq n \leq N. \end{cases}$$

From the hypotheses, this operator T is continuous, and for $y \in \mathcal{S}$, in view of $0 \leq h < 1$, we have

$$Ty_n \leq \begin{cases} 3 + h + K \displaystyle\sum_{s=N}^{n-1} \frac{1}{p_s} \sum_{\eta=s}^{\infty} q_{\eta+1} \leq 3 + h + \frac{1-h}{4} < 4, & n \geq N, \\ 3 + h < 4, & N - m \leq n \leq N, \end{cases}$$

and, similarly

$$Ty_n \geq 3(1-h), \qquad n \geq N - m.$$

Thus, $T(\mathcal{S}) \subseteq \mathcal{S}$. Therefore, by the Schauder fixed-point theorem, T has a fixed-point $y \in \mathcal{S}$. It is clear that $y = \{y_n\}$ is a positive solution of (1).

Now assume that $h > 1$. Let $K = \max\{|f(u)| : 2(h-1) \le u \le 4h\}$. Choose a positive integer $N \ge n_0$ sufficiently large, so that

$$K \sum_{n=N}^{\infty} \frac{1}{p_n} \sum_{s=n}^{\infty} q_{s+1} \le \frac{h(h-1)}{4}.$$

Let B_N be as above, and let

$$\mathcal{S} = \{y \in B_N : 2(h-1) \le y_n \le 4h,\ n \ge N - m\}.$$

Again, $\mathcal{S}$ is a bounded, closed, and convex subset of B_N. Define an operator $T : \mathcal{S} \to B_N$ as follows:

$$Ty_n = \begin{cases} 3h + 1 - \dfrac{1}{h} y_{n+k} + \dfrac{1}{h} \displaystyle\sum_{s=N}^{n+k-1} \frac{1}{p_s} \sum_{\eta=s}^{\infty} q_{\eta+1} f(y_{\eta+1-\ell}), & n \ge N, \\ Ty_N, & N - m \le n \le N. \end{cases}$$

For this continuous operator also, it is easy to see that $T(\mathcal{S}) \subseteq \mathcal{S}$, and hence, by the Schauder fixed-point theorem, T has a fixed point $y \in \mathcal{S}$. Once again, it is clear that this $y = \{y_n\}$ is a positive solution of (1).

Thus, we have established the existence of positive solutions for equation (1), when $h \ge 0$, $h \ne 1$. Next, we shall show that $M^+ \ne \emptyset$. For this, suppose that $\{y_n\} \in$ WOS. Let $n_1 \ge n_0 \in Z$ be such that $y_n > 0$, $y_{n-m} > 0$ for all $n \ge n_1$. Let $z_n = y_n + hy_{n-k}$. Then, we have $\Delta z_n = \Delta y_n + h\Delta y_{n-k}$, $\Delta z_{n+1} = \Delta y_{n+1} + h\Delta y_{n+1-k}$, and

$$\Delta z_n \Delta z_{n+1} = \Delta y_n \Delta y_{n+1} + h\left(\Delta y_n \Delta y_{n+1-k} + \Delta y_{n+1} \Delta y_{n-k}\right) + h^2 \Delta y_{n-k} \Delta y_{n+1-k}.$$

Thus, in view of (i) and (ii), we find that $\Delta z_n \Delta z_{n+1} \le 0$, and hence, $\{\Delta z_n\}$ is oscillatory. Define $w_n = p_n \Delta z_n$, so that $\{w_n\}$ is also oscillatory. On the other hand, from (1), we have $\Delta w_n = -q_{n+1} f(y_{n+1-\ell})$, $n \ge n_1$, and hence, (iii) implies that $\Delta w_n \le 0$, and so w_n is nonincreasing. This contradiction shows that $\{y_n\} \notin$ WOS. Also, from [1, Theorem 3(b)], we have $\{y_n\} \notin M^-$. Thus, $\{y_n\} \in M^+$.

EXAMPLE 1. Consider the difference equation

$$\Delta\left(\frac{1}{n}\Delta\left(y_n + 2y_{n-2}\right)\right) + \frac{3}{(n-2)^3 n(n+1)} y_{n-2}^3 = 0, \qquad n \ge 3, \tag{2}$$

for which all the conditions of Theorem 1 are satisfied. Thus, from the proof of Theorem 1, it follows that (2) has a solution $\{y_n\}$ in the class M^+ such that $2 \le y_n \le 8$ for all sufficiently large n. It also has an unbounded solution $\{y_n\} = \{n\}$ which also belongs to the class M^+.

THEOREM 2. *With respect to the difference equation (1), assume that in addition to the condition (iii), the following hold:*

(vi) *k is an odd positive integer,*

(vii) *$h_n \equiv h \le 0$ and $\ne -1$ for all $n \in Z$, and*

(viii) $\sum_{n=n_0}^{\infty} (1/p_n) \sum_{s=n_0}^{n-1} q_{s+1} < \infty$.

Then, $M^+ \bigcup M^- \ne \emptyset$.

PROOF. We shall prove the existence of positive solutions in the class $M^+ \bigcup M^-$; the existence of negative solutions in $M^+ \bigcup M^-$ is similar.

First, assume that $-1 < h \le 0$. Let $K = \max\{|f(u)| : 2(1+h)/3 \le u \le 4/3\}$, and $N \ge n_0$ sufficiently large, so that

$$K \sum_{n=N}^{\infty} \frac{1}{p_n} \sum_{s=N}^{n-1} q_{s+1} \le \frac{1+h}{3}.$$

Let the space B_N be as in Theorem 1. We define a subset $\mathcal{S}$ of B_N as

$$\mathcal{S} = \left\{ y \in B_N : \frac{2(1+h)}{3} \le y_n \le \frac{4}{3},\ n \ge N-m \right\}.$$

Clearly, $\mathcal{S}$ is a bounded, closed, and convex subset of B_N. Now, we define an operator $T : \mathcal{S} \to B_N$ as follows:

$$Ty_n = \begin{cases} 1 + h - hy_{n-k} + \displaystyle\sum_{s=n}^{\infty} \frac{1}{p_s} \sum_{\eta=N}^{s-1} q_{\eta+1} f(y_{\eta+1-\ell}), & n \ge N, \\ Ty_N, & N-m \le n \le N. \end{cases}$$

This operator T is continuous, and it is easy to see that $T(\mathcal{S}) \subseteq \mathcal{S}$, and hence, by the Schauder fixed-point theorem, T has a fixed point $y \in \mathcal{S}$. This fixed point is a positive solution of (1).

Now assume that $h < -1$. Let $K = \max\{|f(u)| : -h/2 \le u \le -2h\}$. Let $N \ge n_0$ be so large that

$$-\frac{K}{h} \sum_{n=N}^{\infty} \frac{1}{p_n} \sum_{s=N}^{n-1} q_{s+1} \le -\frac{(1+h)}{4}.$$

Let B_N be as above, and let

$$\mathcal{S} = \left\{ y \in B_N : -\frac{h}{2} \le y_n \le -2h,\ n \ge N-m \right\}.$$

Again, $\mathcal{S}$ is a bounded, closed, and convex subset of B_N. Define an operator $T : \mathcal{S} \to B_N$ as follows:

$$Ty_n = \begin{cases} -h - 1 - \dfrac{1}{h} y_{n+k} + \dfrac{1}{h} \displaystyle\sum_{s=n+k}^{\infty} \frac{1}{p_s} \sum_{\eta=N}^{s-1} q_{\eta+1} f(y_{\eta+1-\ell}), & n \ge N, \\ Ty_N, & N-m \le n \le N. \end{cases}$$

For this continuous operator also, it is easy to see that $T(\mathcal{S}) \subseteq \mathcal{S}$, and hence, by the Schauder fixed-point theorem, T has a fixed point $y \in \mathcal{S}$. Once again, it is clear that this fixed point is a positive solution of (1).

This proves the existence of positive solutions $\{y_n\}$ for equation (1), when $h \le 0$, $h \ne -1$. Further, from [1, Theorem 5], we have $\{y_n\} \notin$ WOS. Thus, in conclusion, we find that $\{y_n\} \in M^+ \bigcup M^-$.

EXAMPLE 2. For the difference equations

$$\Delta \left(4^n \Delta \left(y_n - 4y_{n-1}\right)\right) + 7\left(4^{5-n}\right) y_{n-2}^5 = 0, \qquad n \ge 3, \tag{3}$$

and

$$\Delta \left(4^n \Delta \left(y_n - \frac{1}{4} y_{n-1}\right)\right) + 2^{4(n-2)/3} y_{n-2}^{1/3} = 0, \qquad n \ge 3, \tag{4}$$

all the conditions of Theorem 2 are satisfied. In fact, $\{y_n\} = \{2^n\} \in M^+$ is a solution of (3), and $\{y_n\} = \{2^{-n}\} \in M^-$ is a solution of (4).

THEOREM 3. *With respect to the difference equation (1), assume that in addition to conditions (i), (ii), (iii), and (viii), the following holds:*

(ix) $\lim_{n\to\infty} \sum_{s=n_0}^{n-1} q_{s+1} = \infty$.

Then, $M^- \ne \emptyset$.

PROOF. Once again, we shall prove the existence of positive solutions of (1) in the class M^-; the existence of negative solutions in M^- follows similarly.

First, assume that $0 \le h < 1$. Let $K = \max\{|f(u)| : 5(1-h)/2 \le u \le 4\}$, and let $N \ge n_0$ be such that

$$K \sum_{n=N}^{\infty} \frac{1}{p_n} \sum_{s=N}^{n-1} q_{s+1} \le \frac{1-h}{2}.$$

Let B_N be as before, and $\mathcal{S} \subset B_N$ be defined as

$$\mathcal{S} = \left\{ y \in B_N : \frac{5}{2}(1-h) \le y_n \le 4,\ n \ge N-m \right\}.$$

Clearly, $\mathcal{S}$ is a bounded, closed, and convex subset of B_N. Define an operator $T : \mathcal{S} \to B_N$ as follows:

$$Ty_n = \begin{cases} 3 + h - hy_{n-k} - \displaystyle\sum_{s=N}^{n-1} \frac{1}{p_s} \sum_{\eta=N}^{s-1} q_{\eta+1} f(y_{\eta+1-\ell}), & n \ge N, \\ Ty_N = 3 + h - hy_{N-k}, & N - m \le n \le N. \end{cases}$$

This operator T is continuous, and as earlier, it is easy to see that $T(\mathcal{S}) \subseteq \mathcal{S}$. Therefore, by the Schauder fixed-point theorem, T has a fixed point $y \in \mathcal{S}$. It is clear that $y = \{y_n\}$ is a positive solution of (1).

Now assume that $h > 1$. Let $K = \max\{|f(u)| : 5(h-1)/2 \le u \le 4h\}$, and let $N \ge n_0$ be such that

$$K \sum_{n=N}^{\infty} \frac{1}{p_n} \sum_{s=N}^{n-1} q_{s+1} \le \frac{h(h-1)}{2}.$$

Let B_N be as above, and

$$\mathcal{S} = \left\{ y \in B_N : \frac{5(h-1)}{2} \le y_n \le 4h,\ n \ge N-m \right\}.$$

Again, $\mathcal{S}$ is a bounded, closed, and convex subset of B_N. Define an operator $T : \mathcal{S} \to B_N$ as follows:

$$Ty_n = \begin{cases} 3h + 1 - \dfrac{1}{h} y_{n+k} - \dfrac{1}{h} \displaystyle\sum_{s=N}^{n+k-1} \frac{1}{p_s} \sum_{\eta=N}^{s-1} q_{\eta+1} f(y_{\eta+1-\ell}), & n \ge N, \\ Ty_N, & N - m \le n \le N. \end{cases}$$

For this continuous operator also, it is easy to see that $T(\mathcal{S}) \subseteq \mathcal{S}$, and hence, by the Schauder fixed-point theorem, T has a fixed point $y \in \mathcal{S}$. Once again, it is clear that this $y = \{y_n\}$ is a positive solution of (1).

Thus, we have obtained the existence of positive solutions for equation (1), when $h \ge 0$, $h \ne 1$. From the proof of Theorem 1, $\{y_n\} \notin$ WOS, and also from [1, Theorem 1], $\{y_n\} \notin M^+$. Hence, $\{y_n\} \in M^-$.

EXAMPLE 3. Consider the difference equation

$$\Delta\left(4^n \Delta\left(y_n + 4y_{n-2}\right)\right) + 17\left(2^{8(n-1)/5}\right) y_{n-1}^{3/5} = 0, \qquad n \ge 2, \tag{5}$$

for which all the conditions of Theorem 3 are satisfied. In fact, the sequence $\{y_n\} = \{2^{-n}\}$ is a solution of (5) which belongs to the class M^-.

THEOREM 4. *With respect to the difference equation (1), assume that in addition to condition (iv), the following hold:*

(x) $h_n \equiv h \geq 0$ *for all* $n \geq n_0 \in Z$, *and*

(xi) $\limsup_{n\to\infty} \sum_{s=N}^{n-1} q_{s+1} = \infty$.

Then, WOS $\neq \emptyset$.

PROOF. Suppose the result is not true, then there exists a solution $\{y_n\}$ of (1) such that eventually, either

$$y_n > 0 \quad \text{and} \quad \Delta y_n > 0, \quad \text{or} \tag{6}$$

$$y_n > 0 \quad \text{and} \quad \Delta y_n < 0, \quad \text{for all } n \geq N - m \tag{7}$$

holds. Assume that (6) holds. Let $z_n = y_n + hy_{n-k}$, so that equation (1) can be written as $\Delta(p_n \Delta z_n) = -q_{n+1} f(y_{n+1-\ell})$. Let $w_n = (p_n \Delta z_n)/f(y_{n+1-\ell})$, $n \geq N$. Then, $w_n > 0$ and the condition (c_3) implies that $\Delta w_n \leq -q_{n+1}$, $n \geq N$. Now, summing the last inequality from N to $n-1$ with N sufficiently large, we obtain

$$w_n - w_N \leq -\sum_{s=N}^{n-1} q_{s+1}.$$

In the above inequality, the right side, in view of condition (xi), tends to $-\infty$. But, this contradicts the fact that $w_n > 0$.

Next assume that (7) holds. From condition (xi), we find that there exists an integer $N_1 \geq N$ such that

$$\sum_{s=N_1}^{n-1} q_{s+1} > 0. \tag{8}$$

For this, we set $F_n = \sum_{s=N}^{n-1} q_{s+1}$ and define $N_1 = \sup\{n \geq N : F_n = 0\}$. Clearly, $F_{N_1} = 0$ and $F_n > 0$ for $n > N_1$. Then, $\sum_{s=N_1}^{n-1} q_{s+1} = F_n - F_{N_1} = F_n > 0$. Now from (7), (8), and Abel's transformation [2, p. 35], it follows for $n > N_1$ that

$$\sum_{s=N_1}^{n-1} q_{s+1} f(y_{s+1-\ell}) = f(y_{n+1-\ell}) \sum_{s=N_1}^{n-1} q_{s+1} - \sum_{s=N_1}^{n-1} \Delta f(y_{s+1-\ell}) \left(\sum_{\tau=N_1}^{s-1} q_{\tau+1} \right) > 0.$$

Thus, on summing (1) from N_1 to $n-1$, we get

$$p_n \Delta z_n - p_{N_1} \Delta z_{N_1} = -\sum_{s=N_1}^{n-1} q_{s+1} f(y_{s+1-\ell}) < 0.$$

Hence, $p_n \Delta z_n < p_{N_1} \Delta z_{N_1}$. It is clear that $\Delta z_{N_1} < 0$. Therefore, $\Delta z_n < p_{N_1} \times \Delta z_{N_1}/p_n$. Again summing from N_1 to $n-1$, we obtain

$$z_n - z_{N_1} < p_{N_1} \Delta z_{N_1} \sum_{s=N_1}^{n-1} \frac{1}{p_s},$$

which in view of (iv), implies that $z_n \to -\infty$ as $n \to \infty$. But, by (7), we have $z_n > 0$. This contradiction completes the proof.

EXAMPLE 4. Consider the difference equation

$$\Delta\left(n\Delta\left(y_n + 2y_{n-2}\right)\right) + \frac{(-1)^n(12n+6)}{\left(2-(-1)^n\right)^3} y_{n-1}^3 = 0, \qquad n \geq 2, \tag{9}$$

for which all the conditions of Theorem 4 are satisfied. In fact, $\{y_n\} = \{2 + (-1)^n\} \in$ WOS is a solution of (9).

THEOREM 5. *With respect to the difference equation (1), assume that the conditions (i), (iii), (iv), (x), and (xi) are satisfied. Then,* OS $\neq \emptyset$.

PROOF. The result follows from the proof of Theorem 1 and [1, Theorems 1 and 3(b)].

REMARK 1. For the existence of solutions of (1) in the class OS when $-1 \leq h_n \equiv h \leq 0$, see [1, Theorem 8]. We also remark that a special case of Theorem 4 has been proved recently in [3].

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